2025未来科学家暑期研究计划 优秀学术海报展

2025 USTC Future Scientist Student Exchange Program Summer Research Excellent Academic Poster Exhibition

基本信息 Basic Information

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Bakry–Émery Curvature Sharpness of Strongly Regular Graphs

1 · Introduction

A **strongly regular graph** SRG(n, d, α, β) is a d-regular graph in which every two adjacent vertices have exactly α common neighbours and every two non-adjacent vertices have exactly β. Its girth is 3 precisely when α ≥ 1.

The **Bakry–Émery curvature** K(∞) is the largest K for which CD(K, ∞) holds at a vertex, a discrete analogue of a Ricci lower bound that depends only on the 2-ball around the vertex.

Conjecture 11.8 (Cushing–Liu–Peyerimhoff, 2016). every girth-3 SRG satisfies CD(2, ∞). Proved for β ≥ α; the case **α > β is open** — the target of this project.

Intuitively, positive Bakry–Émery curvature is closely related to the local structure of neighbourhoods, making the girth-3 case a natural setting to study positive curvature bounds.

Strongly regular graphs are the **diameter-2 distance-regular graphs**, a rich testing ground where the whole 2-ball is encoded in just four parameters, making the curvature computable in closed form yet far from trivial.

2 · Reduction to eigenvalues

$$\mathcal{K}_{G,x}(\infty) = 2 + \frac{\alpha}{2} + \left(\frac{2d(\beta-2)-\alpha^2}{2\beta} + \frac{2}{\beta}\min_{\lambda\in\sigma(A_{S_1(x)}|_{1^\perp})}(\lambda-\tfrac{\alpha}{2})^2\right)_-$$

$$(\mathrm{S}):\ \min_{\lambda}(\lambda-\tfrac{\alpha}{2})^2 \geq \mathrm{RHS}_{\mathrm{sharp}} := \frac{\alpha^2}{4} + d - \frac{\beta d}{2} \iff \mathcal{K}(\infty) = 2 + \frac{\alpha}{2}$$

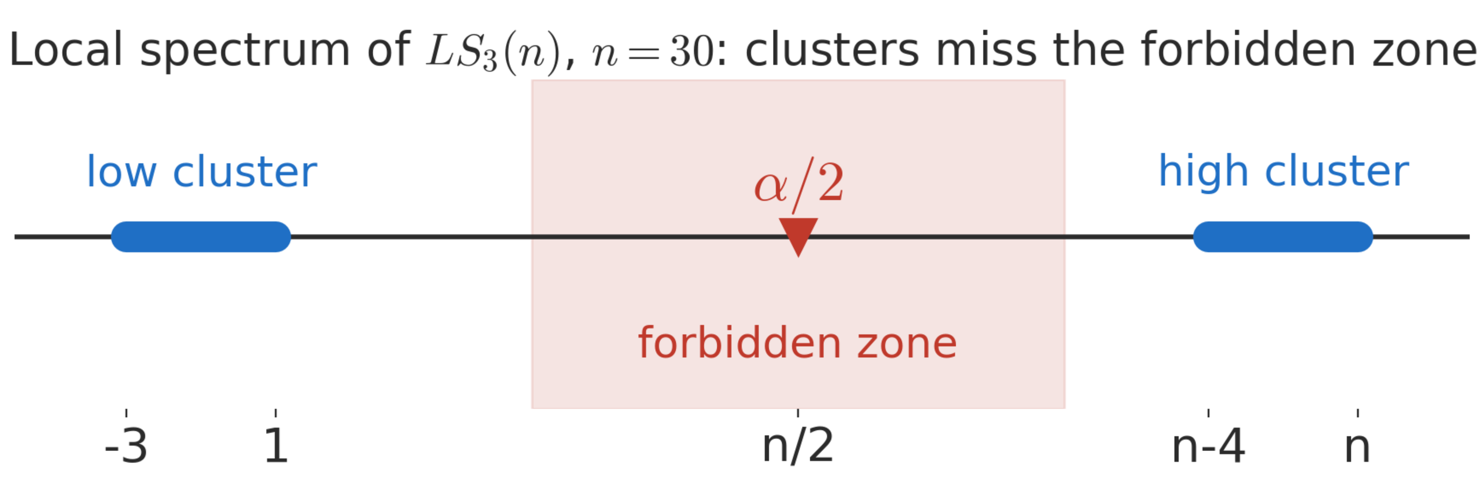
Established. By Theorem 4.3, the computation of K(∞) reduces to the spectrum of the local graph $S_1(x)$. The α-regular graph induced on the d neighbours of x. CD(2,∞) holds exactly when the eigenvalues of $S_1(x)$ stay far enough from α/2.

Where the bracket $\mathrm{RHS}_{\mathrm{sharp}}$ is ≤ 0 the inequality is automatic (the left side is a square), so the real work lies where $\mathrm{RHS}_{\mathrm{sharp}} > 0$.

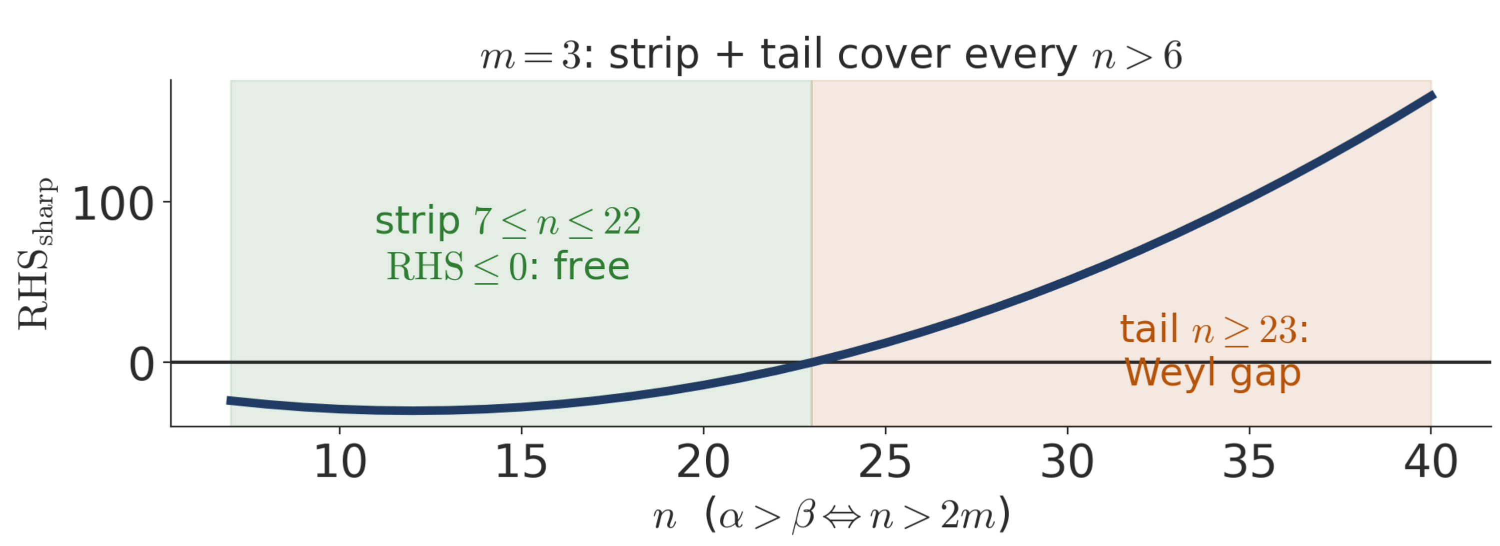
3 · Idea of proof

The local graph of a Latin-square graph LS_m(n) is m cliques K_{n-1} (the lines through x) plus a cross-matrix C whose row sums (m−1)(m−2) are **forced by the parameters alone**, independent of which Latin square is used.

By Weyl's inequality the spectrum splits into two clusters, near −1 and n−2 — both bounded away from α/2:

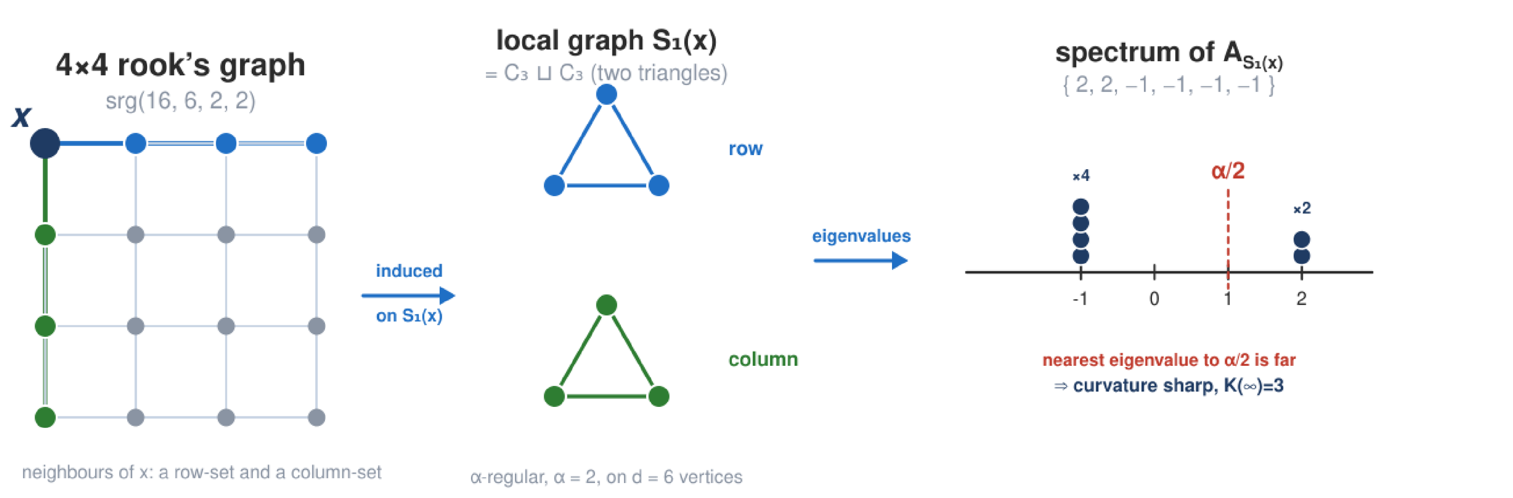


An **automatic strip** handles small n; in the tail, D² − RHS_sharp is linear in n with positive slope, hence positive for every n:



Verified. at the tightest tail points (m=3, n=23; m=4, n=71, varying MOLS): the bound needed 6.25, the actual margin was 870.25.

4 · The objects behind K(∞)



For the 4×4 rook's graph, the six neighbours of a vertex split into a row-triangle and a column-triangle, so the local graph is two disjoint triangles with spectrum {2, 2, −1, −1, −1, −1}.

Established. Theorem 4.3 turns the eigenvalue nearest α/2 into K(∞). Here that eigenvalue is far from α/2 = 1, giving the sharp value K(∞) = 3.

5 · Strategy: classify, then bound

Established. when α > β the graph cannot be a conference graph (those have α = β − 1), so its eigenvalues r, s are integers and Neumaier's 1979 theorem applies: every primitive SRG with smallest eigenvalue −m is a Latin-square graph, a Steiner block graph, or one of finitely many exceptions. We prove the conjecture **one family at a time.**

Concretely, the **Latin-square graphs** come from mutually orthogonal Latin squares and the **Steiner graphs** from block designs; only finitely many sporadic exceptions lie outside these two infinite families, so each admits its own uniform argument.

The payoff is that a single spectral estimate, applied family by family, can settle infinitely many graphs at once rather than checking cases one graph at a time.

6 · Main result

Theorem. every Latin-square graph $LS_m(n)$ with n > 2m — for any choice of the underlying Latin squares is ∞-curvature sharp, K(∞) = 2 + α/2. Hence all Latin-square graphs with α > β satisfy CD(2, ∞).

This is **stronger than the conjecture asks**: it pins the curvature to its exact maximal value, and holds **uniformly over all Latin squares of a given order**, not just special ones.

8 · Conclusion

The Latin-square branch of the α > β case is complete. What remains:

- the Steiner block-graph branch, where the same clique decomposition applies;
- the finitely many Neumaier exceptions per smallest eigenvalue −m;
- Problem 11.15: classify all ∞-curvature-sharp SRGs.

The longer-term goal is a **full classification of the curvature-sharp SRGs**, with distance-regular graphs of larger diameter as the natural horizon.

Two features make the Latin-square argument likely to transfer: the local graph is a union of cliques, and the cross-adjacency counts are **fixed by the parameters, not the specific design**. The same clique-plus-Weyl estimate should therefore reach the Steiner family, and with the Terwilliger algebra the higher-diameter distance-regular case.

References

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[3] Neumaier, Arch. Math. 33 (1979), 392–400.

[4] Brouwer, Van Maldeghem, Strongly Regular Graphs, CUP (2022).